Approximation by Fourier sums and interpolation trigonometric polynomials in classes of differentiable functions with high exponents of smoothness

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Let C and $L_p,\ 1 \le p \le \infty$, be the spaces of 2π–periodic functions with the standart norms $\|\cdot\|_C$ and $\|\cdot\|_p$. Further, let $W^r_{\beta,p}, 1 \le p \le \infty$, be the sets of all 2π-periodic functions f, representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(x-t) B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R},\ \varphi \perp 1,\ \|\varphi\|_p \le 1, \tag{1}$$

where $B_{r,\beta}(\cdot)$ are Weyl-Nagy kernels of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r}\cos\left(kt - \frac{\beta\pi}{2}\right), \quad r > 0, \quad \beta \in \mathbb{R}. \tag{2}$$

The classes $W^r_{\beta,p}$ are called as Weyl-Nagy classes (see, e.g., [1]). If $r \in \mathbb{N}$ and $\beta = r$, then the functions of the form (2) are the well-known Bernoulli kernels and the classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p, which consist of 2π-periodic functions with absolutely continuous derivatives up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_p \le 1$ and $f^{(r)}(x) = \varphi(x)$ for almost everywhere $x \in \mathbb{R}$, where φ is the function from (1).

For arbitrary $\mathfrak{N} \subset X$, where $X = C$ or $L_p,\ 1 \le p \le \infty$, we consider the quantity

$$\varepsilon_n(\mathfrak{N})_X = \sup_{f \in \mathfrak{N}} \|f(\cdot) - S_{n-1}(f;\cdot)\|_X, \tag{3}$$

where $S_{n-1}(f;x)$ is the partial Fourier sum of order $n-1$ of the function f.

In the case of Weyl-Nagy classes $W^r_{\beta,\infty}$ and $X = C$ for the exact upper bounds (3) the following asymptotic estimate holds

$$\varepsilon_n(W^r_{\beta,\infty})_C = \frac{4}{\pi^2}\frac{\ln n}{n^r} + O\left(\frac{1}{n^r}\right), \quad r > 0, \quad \beta \in \mathbb{R}. \tag{4}$$

For $r \in \mathbb{N}$ and $\beta = r$ this estimate was obtained by A.N. Kolmogorov (1935), for arbitrary $r > 0$ by V.T. Pinkevich (1940) and S.M. Nikol'skii (1941). In the general case the estimate (4) follows from results, which were obtained in the works of A.V.Efimov (1960) and S.A. Telyakovskii (1961). It should be also noticed, that a similar asymptotic equality holds for the classes $W^r_{\beta,1}$ in the metric of the space L_1. In these works the parameters r and β of the Weyl-Nagy classes were assumed to be fixed, and the question about the dependence of the remainder term in the estimate (4) on these parameters was not considered.

The character of the dependence on r and β of the remainder term in estimate (4) was investigated by I.G. Sokolov (1955), S.G. Selivanova (1955), G.I. Natanson (1961), S.A. Telyakovskii (1968, 1989) and S.B. Stechkin (1980). In the work of S.B. Stechkin [2] the asymptotic behavior, as $n \to \infty$ and $r \to \infty$, of the quantities $\varepsilon_n(W^r_{\beta,\infty})_C$ was completely investigated. Besides, S.B. Stechkin [2, theorem 4] proved that for rapidly growing r the remainder can be improved. Namely, for arbitrary $r \ge n+1$

and $\beta \in \mathbb{R}$ the following equality holds:

$$\varepsilon_n(W^r_{\beta,\infty})_C = \frac{1}{n^r}\left(\frac{4}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{5}$$

where $O(1)$ is a quantity uniformly bounded with respect to n, r and β. If $r/n \to \infty$, then the estimate (5) becomes the asymptotic equality. It also follows from [2] that for the quantity $\varepsilon_n(W^r_{\beta,1})_{L_1}$ the analogous estimate to (5) takes place. S.A. Telyakovskii (1989) showed that the remainder in formulas (5) can be replaced by a smaller one, namely, write $O(1)(1+2/n)^{-r}$ instead of $O(1)(1+1/n)^{-r}$.

We establish generalized analogs of estimates (5) for quantities $\varepsilon_n(W^r_{\beta,p})_C$ and $\varepsilon_n(W^r_{\beta,1})_{L_p}$, respectively, for arbitrary values $1 \le p \le \infty$.

Theorem 1. *Let $1 \le p \le \infty$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Then for $r \ge n+1$ the following estimates hold:*

$$\varepsilon_n(W^r_{\beta,p})_C = \frac{1}{n^r}\left(\frac{\|\cos t\|_{p'}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{6}$$

$$\varepsilon_n(W^r_{\beta,1})_{L_p} = \frac{1}{n^r}\left(\frac{\|\cos t\|_{p}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{7}$$

where $1/p + 1/p' = 1$ and $O(1)$ are quantities uniformly bounded in all analyzed parameters. The estimates (6) and (7) are the asymptotic equalities, as $r/n \to \infty$.

Let $f \in C$. By $\tilde{S}_{n-1}(f;x)$ we denote a trigonometric polynomial of degree $n-1$, that interpolates $f(x)$ at the equidistant nodes $x_k^{(n-1)} = \frac{2k\pi}{2n-1}$, $k \in \mathbb{Z}$, i.e., such that $\tilde{S}_{n-1}(f; x_k^{(n-1)}) = f(x_k^{(n-1)})$, $k \in \mathbb{Z}$.

For $\mathfrak{N} \subset C$ and $X = C$ or $X = L_p, 1 \le p \le \infty$, consider the following approximative characteristic

$$\tilde{\varepsilon}_n(\mathfrak{N})_X = \sup_{f \in \mathfrak{N}} \|f(\cdot) - \tilde{S}_{n-1}(f;\cdot)\|_X. \tag{8}$$

The problems of finding of asymptotic behavior for quantity of the form (8) in important functional classes $\mathfrak{N}$ was investigated by S.M. Nikol'skii, V.P. Motornyi, A.I. Stepanets, A.S. Serdyuk, and others.

The following statement is true [3].

Theorem 2. *Let $1 \le p \le \infty$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Then for $r \ge n+1$ the following estimates hold:*

$$\tilde{\varepsilon}_n(W^r_{\beta,p})_C = \frac{1}{n^r}\left(\frac{2}{\pi}\|\cos t\|_{p'} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \tag{9}$$

$$\tilde{\varepsilon}_n(W^r_{\beta,1})_{L_p} = \frac{1}{n^r}\left(\frac{2^{1-\frac{1}{p}}}{\pi^{1+\frac{1}{p}}}\|\cos t\|_p^2 + O(1)\left(\frac{1}{n} + \left(1+\frac{1}{n}\right)^{-r}\right)\right), \tag{10}$$

where$1/p + 1/p' = 1$ and $O(1)$ are quantities uniformly bounded in all analyzed parameters. The estimates (9) and (10) are the asymptotic equalities, as $r/n \to \infty$, $n \to \infty$.

Comparing formulas (6), (7), (9) and (10) we see that

$$\tilde{\varepsilon}_n(W^r_{\beta,p})_C \sim 2\varepsilon_n(W^r_{\beta,p})_C,$$

$$\tilde{\varepsilon}_n(W^r_{\beta,1})_{L_p} \sim \frac{2^{1-\frac{1}{p}}}{\pi^{\frac{1}{p}}}\|\cos t\|_p \ \varepsilon_n(W^r_{\beta,1})_{L_p},$$

as $r/n \to \infty$, $n \to \infty$.

References

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